

## 32.14 VECTOR SPACE

(R.G.P.V., Bhopal, III Semester, Dec. 2007)

Let  $V$  be a non-empty set and  $F$  be the field of real numbers. Let we have two compositions one is plus (+) between two members of  $V$  and other is dot ( $\cdot$ ) between a member of  $V$  and member of  $F$ .  $V$  is said to be vector space if the following properties hold good.

### 1. Closure property.

$$\forall a, b \in R \Rightarrow a + b \in R.$$

### 2. Associativity of addition.

For all  $\alpha, \beta, \gamma \in V$ .

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma.$$

### 3. Existence of the neutral element.

There exists an element  $0 \in V$ , such that

$$\alpha + 0 = 0 + \alpha = \alpha, \text{ for all } \alpha \in V.$$

### 4. Existence of additive inverse.

For each  $\alpha \in V$ , there exists  $\beta \in V$ , such that

$$\alpha + \beta = \beta + \alpha = 0.$$

### 5. Commutativity of addition.

### Axioms

For all  $\alpha, \beta \in V$ ,  $\alpha + \beta = \beta + \alpha$

### 6. Closure property.

$\forall \alpha, \beta \in R \Rightarrow \alpha \cdot \beta \in R$

### 7. Associativity of scalar multiplication.

For all  $x, y \in F$  and  $\alpha \in V$

$$x(y\alpha) = (xy)\alpha.$$

### 8. Distributivity of scalar multiplication over addition.

For all  $x \in F$ ,  $\alpha, \beta \in V$ ,

$$x(\alpha + \beta) = x\alpha + x\beta.$$

### 9. Distributivity of scalar multiplication over addition in $F$ .

For all  $x, y \in F$ ,  $\alpha \in V$ .

$$(x+y)\alpha = x\alpha + y\alpha.$$

### 10. Property of unity.

If 1 be the identity in  $F$ , then for all  $\alpha \in V$ .

$$1 \cdot \alpha = \alpha.$$

Note. Vectors will be denoted by  $\alpha, \beta, \gamma$  while scalars will be denoted by  $a, b, c, d$  or  $x, y, z$ .

**Theorem 1.** Let  $V(F)$  be a vector space, and

(i) If  $\alpha$  is a non-zero element of  $V$  and  $a, b \in F$ , then

$$a\alpha = b\alpha \Rightarrow a = b$$

(ii) If  $a$  is a non-zero element of  $F$  and  $\alpha, \beta \in V$ , then

$$a\alpha = a\beta \Rightarrow \alpha = \beta.$$

**Proof.** (i) We have,

$$a\alpha = b\alpha \Rightarrow a\alpha - b\alpha = \bar{0} \Rightarrow (a-b)\alpha = \bar{0}.$$

$$\Rightarrow a-b = 0 \quad [\because \bar{\alpha} \neq 0] \Rightarrow a = b.$$

Hence,

$$a\alpha = b\alpha \Rightarrow a = b$$

$$[\alpha \neq \bar{0}]$$

(ii) We have,

$$a\alpha = a\beta \Rightarrow a\alpha - a\beta = \bar{0} \Rightarrow a(\alpha - \beta) = \bar{0}.$$

$$\Rightarrow \alpha - \beta = \bar{0} \quad [\because a \neq 0] \Rightarrow \alpha = \beta.$$

Hence,

$$a\alpha = a\beta \Rightarrow \alpha = \beta$$

$$[a \neq 0]$$

### 33.19 LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

Vectors (matrices)  $X_1, X_2, \dots, X_n$  are said to be dependent if

- (1) all the vectors (row or column matrices) are of the same order.
- (2)  $n$  scalars  $\lambda_1, \lambda_2, \dots, \lambda_n$  (not all zero) exist such that

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \dots + \lambda_n X_n = 0$$

Otherwise they are linearly independent.

**Example 4.** Show that the vectors  $X_1 = (1, 2, 3)$ ,  $X_2 = (3, -1, 4)$  and  $X_3 = (4, 1, 7)$  are linearly dependent.

**Solution.**  $X_1 = (1, 2, 3)$

$$X_2 = (3, -1, 4)$$

$$X_3 = (4, 1, 7)$$

If  $\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = 0$

$$\Rightarrow \lambda_1(1, 2, 3) + \lambda_2(3, -1, 4) + \lambda_3(4, 1, 7) = 0$$

$$\Rightarrow [(\lambda_1 + 3\lambda_2 + 4\lambda_3), (2\lambda_1 - \lambda_2 + \lambda_3), (3\lambda_1 + 4\lambda_2 + 7\lambda_3)] = (0, 0, 0)$$

$$\Rightarrow \lambda_1 + 3\lambda_2 + 4\lambda_3 = 0 \quad \dots (1)$$

$$2\lambda_1 - \lambda_2 + \lambda_3 = 0 \quad \dots (2)$$

$$3\lambda_1 + 4\lambda_2 + 7\lambda_3 = 0 \quad \dots (3)$$

Solving (1), (2) and (3), we get

$$\lambda_1 = 1, \lambda_2 = 1 \text{ and } \lambda_3 = 1$$

So, the vectors are linearly dependent.

**Proved.**

**Example 5.** Are the vectors

$X_1 = (1, 0, 0)$ ,  $X_2 = (0, 1, 0)$  and  $X_3 = (0, 0, 1)$  linearly dependent?

**Solution.** Consider the matrix equation  $\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = 0$

$$\Rightarrow \lambda_1(1, 0, 0) + \lambda_2(0, 1, 0) + \lambda_3(0, 0, 1) = 0$$

$$\Rightarrow (\lambda_1 + 0\lambda_2 + 0\lambda_3, 0\lambda_1 + \lambda_2 + 0\lambda_3, 0\lambda_1 + 0\lambda_2 + \lambda_3) = (0, 0, 0)$$

$$\Rightarrow (\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0)$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$$

As  $\lambda_1, \lambda_2, \lambda_3$  all are zero, therefore  $X_1, X_2, X_3$  are linearly independent vectors. **Ans.**

**Example 6.** Examine the following vectors for linear dependence and find the relation exists

$$X_1 = (1, 2, 4), \quad X_2 = (2, -1, 3), \quad X_3 = (0, 1, 2), \quad X_4 = (-3, 7, 2)$$

**Solution.** Consider the matrix equation

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4 = 0$$

$$\Rightarrow \lambda_1(1, 2, 4) + \lambda_2(2, -1, 3) + \lambda_3(0, 1, 2) + \lambda_4(-3, 7, 2) = 0$$

$$\lambda_1 + 2\lambda_2 + 0\lambda_3 - 3\lambda_4 = 0$$

$$2\lambda_1 - \lambda_2 + \lambda_3 + 7\lambda_4 = 0$$

$$4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 = 0$$

This is the homogeneous system

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad A \lambda = 0$$

$$R_2 \rightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 - 4R_1$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 + 2\lambda_2 - 3\lambda_4 = 0$$

$$-5\lambda_2 + \lambda_3 + 13\lambda_4 = 0$$

$$\lambda_3 + \lambda_4 = 0$$

Let  $\lambda_4 = t$ ,

$$\lambda_3 + t = 0 \Rightarrow \lambda_3 = -t$$

$$-5\lambda_2 - t + 13t = 0 \Rightarrow \lambda_2 = \frac{12t}{5}$$

$$\lambda_1 + \frac{24t}{5} - 3t = 0 \Rightarrow \lambda_1 = \frac{-9t}{5}$$

Hence, the given vectors are linearly dependent.

Substituting the values of sum in (1), we get

### 33.21 VECTOR SUB SPACES

Let  $V$  be a vector space over a field  $F$ , then, a non-empty subset  $W$  of  $V$  is called a vector subspace of  $V$ , if  $W$  is a vector space in its own right with respect to the addition and scalar multiplication compositions on  $V$ , restricted only on points of  $W$ .

**Remark.** In an arbitrary vector space  $V$ , the sets  $\{0\}$  and  $V$  are clearly subspaces of  $V$  and are known as trivial sub-spaces. However, our interest lies in non-trivial subspaces.

**Example 20.** Let  $R$  be the field of real numbers. Which of the following are subspaces of  $V_3(R)$ ?

- (i)  $W_1 = \{(x, 2y, 3z) : x, y, z \in R\}$
- (ii)  $W_2 = \{(x, x, x) : x \in R\}$
- (iii)  $W_3 = \{(x, y, z) : x, y, z \text{ are rational numbers}\}$

**Solution.** (i) Here  $W_1 = \{(x, 2y, 3z) : x, y, z \in R\}$ .

Let  $\alpha = (x_1, 2y_1, 3z_1)$  and  $\beta = (x_2, 2y_2, 3z_2)$  be any two arbitrary elements of  $W_1$ , then  $x_1, y_1, z_1, x_2, y_2, z_2 \in R$ . If  $a, b \in R$  be any two real numbers, then

$$\begin{aligned} a\alpha + b\beta &= a(x_1, 2y_1, 3z_1) + b(x_2, 2y_2, 3z_2) = (ax_1 + bx_2, 2ay_1 + 2by_2, 3az_1 + 3bz_2) \\ &= [ax_1 + bx_2, 2(ay_1 + by_2), 3(az_1 + bz_2)] \in W_1 \quad [\because ax_1 + bx_2 \text{ etc. } \in R] \end{aligned}$$

$\therefore a, b \in R$  and  $\alpha, \beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1$ .

Hence,  $W_1$  is a subspace of  $V_3(R)$ .

(ii) Here  $W_2 = \{(x, x, x) : x \in R\}$ . Let  $\alpha = (x_1, x_1, x_1)$  and  $\beta = (x_2, x_2, x_2)$  be any two elements of  $W_2$ , then  $x_1, x_2 \in R$ . If  $a, b \in R$  be any two real numbers, then we have

$$\begin{aligned} a\alpha + b\beta &= a(x_1, x_1, x_1) + b(x_2, x_2, x_2) = (ax_1 + bx_2, ax_1 + bx_2, ax_1 + bx_2) \in W_2 \\ &\quad [\because ax_1 + bx_2 \in W_2] \end{aligned}$$

Hence  $W_2$  is a subspace of  $V_3(R)$ .

(iii) Here  $W_3 = \{(x, y, z) : x, y, z \text{ are rational numbers}\}$ .

Let  $\alpha = (4, 5, 7)$  be any element of  $W_3$ . If  $a = \sqrt{6}$  is an element of  $R$ , then

$$a\alpha = \sqrt{6}(4, 5, 7) = (4\sqrt{6}, 5\sqrt{6}, 7\sqrt{6}) \notin W_3. \text{ Since } 4\sqrt{6}, 5\sqrt{6}, 7\sqrt{6} \text{ are not rational numbers.}$$

Consequently,  $W_3$  is not closed with respect to scalar multiplication. Hence,  $W_3$  is not a subspace of  $V_3(R)$ . Ans.

### 33.28 BASIS (R.G.P.V., Bhopal, III Semester, Dec. 2006)

Let  $V$  be a vector space. A collection of vectors  $\alpha_1, \alpha_2, \dots, \alpha_r$  is said to form a basis of  $V$  if  $\alpha_1, \alpha_2, \dots, \alpha_r$  are linearly independent and if they generate  $V$ .

#### Coordinate of a Vector

Let  $V(F)$  be a finite dimensional vector space. Let  $B = \alpha_1, \alpha_2, \dots, \alpha_n$  be ordered basis of  $V$ . Let  $\alpha \in V$ . Then there exists a unique  $n$ -tuple  $(x_1, x_2, \dots, x_n)$  of scalars such that

$$\alpha = x_1 \alpha_1 + x_2 \alpha_2 + \dots + x_n \alpha_n$$

$(x_1, x_2, \dots, x_n)$  is called coordinates of the basis  $V$ .

### 33.29 DIMENSION OR RANK OF A VECTOR SPACE

(R.G.P.V., Bhopal, III Semester, Dec. 2005)

The number of vectors presents in a basis of a vector space  $V$  is called the dimension of  $V$ . It is denoted by  $\dim(V)$ .

**Example 25.** Dimension of the vector space  $V_4$  is 4, since the four vectors  $(1, 0, 0, 0)$ ,  $(0, 1, 0, 0)$ ,  $(0, 0, 1, 0)$ , and  $(0, 0, 0, 1)$  form a basis of  $V_4$ .

**Example 26.**  $\dim(V_n) = n$ , since there are  $n$  number of vectors in a basis of  $V_n$ .

Here, we are mainly concern with finite dimensional vector space. The dimension of vector space may be infinite.

**Example 27.** Each set of  $(n+1)$  or more vectors of a finite dimensional vector space  $V(F)$  of dimension  $n$  is :

(i) linearly dependent

(ii) a basis of  $V(F)$

(iii) a subspace of  $V(F)$

(iv) linearly independent

(R.G.P.V., Bhopal, III Semester, Dec. 2007, 2006)

**Solution.** Dimension of vector space  $V(F)$  is  $n$ , therefore  $V(F)$  may have at most  $n$  independent vectors. Here the number of vectors are  $(n+1)$ , so they are linearly dependent.

Ans.

**Example 28.** Show that the vectors  $(1, 0, 0)$ ,  $(1, 1, 0)$ ,  $(1, 1, 1)$  form a basis for  $R^3$ .

(R. G. P. V. Bhopal, III Semester, June 2007)

**Solution.** Let  $a_1, a_2, a_3 \in R$  be such that

$$a_1(1, 0, 0) + a_2(1, 1, 0) + a_3(1, 1, 1) = 0$$

$$(a_1 + a_2 + a_3) + (0 + a_2 + a_3) + (0 + 0 + a_3) = 0 \quad \dots (1)$$

$$a_1 + a_2 + a_3 = 0$$

$$a_2 + a_3 = 0$$

$$a_3 = 0$$

The matrix of the coefficients of the equations is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Rank}(A) = 3$$

$$a_3 = 0, a_2 = 0 \text{ and } a_1 = 0$$

The non-zero values of  $a_1, a_2, a_3$  do not exist which can satisfy (1).

Thus,  $(1, 0, 0), (1, 1, 0)$  and  $(1, 1, 1)$  are linearly independent.

Hence, the set of given vectors form a basis of  $R^3$ .

**Proved.**

**Example 29.** Show that  $\alpha_1 = (1, 0, 0), \alpha_2 = (0, 1, 0), \alpha_3 = (0, 0, 1)$  form a basis of the vectors space  $V_3$ .

**Solution.** Let  $a_1, a_2, a_3$  be non-zero real numbers.

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0$$

$$a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(0, 0, 1) = 0 \quad \dots (1)$$

$$(a_1, a_2, a_3) = 0$$

$$\Rightarrow a_1 = 0, a_2 = 0, a_3 = 0$$

Thus non-zero values of  $a_1, a_2, a_3$  do not exist which can satisfy (1).

Hence, the given system of vectors is linearly independent.

$$\text{Let } \alpha = x_1(1, 0, 0) + x_2(0, 1, 0) + x_3(0, 0, 1)$$

$$= x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3$$

It shows that any vector space  $V_3$  can be expressed as a linear combination of  $\alpha_1, \alpha_2, \alpha_3$ . So

$\alpha_1, \alpha_2, \alpha_3$  are the generators.

Hence,  $\alpha_1, \alpha_2, \alpha_3$  form basis of  $V$ .

**Proved.**

**Example 30.** Determine whether the following vectors form a basis of  $R^3$  or not

$$(1, 1, 2), (1, 2, 5), (5, 3, 4)$$

**Solution.** We know that  $\dim R^3 = 3$ . Thus if the given set of vectors is linearly independent, then it will be a basis of  $R^3$  otherwise not.

Now, for  $a_1, a_2, a_3 \in R$

$$a_1(1, 1, 2) + a_2(1, 2, 5) + a_3(5, 3, 4) = (0, 0, 0)$$

$$\Rightarrow (a_1 + a_2 + 5a_3, a_1 + 2a_2 + 3a_3, 2a_1 + 5a_2 + 4a_3) = (0, 0, 0)$$

$$\therefore a_1 + a_2 + 5a_3 = 0 \quad \dots (1)$$

$$a_1 + 2a_2 + 3a_3 = 0 \quad \dots (2)$$

$$2a_1 + 5a_2 + 4a_3 = 0 \quad \dots (3)$$

$\therefore$  The coefficient matrix of these equations is

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{bmatrix}$$

$$\text{Here } |A| = 1(8 - 15) - 1(4 - 6) + 5(5 - 4) = -7 + 2 + 5 = 0$$

Rank of  $A \neq 3$

$$a_1 = a_2 = a_3$$

The scalars  $a_1, a_2, a_3$  are not all zero, therefore, the given set  $S$  of vectors is linearly dependent and hence the given set of vectors are not basis set.

Ans.

**Example 32.** Show that the set  $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1), (0, 1, 0)\}$  is not a basis set.

**Solution.** Let  $a_1, a_2, a_3, a_4 \in R$  be such that

$$a_1(1, 0, 0) + a_2(1, 1, 0) + a_3(1, 1, 1) + a_4(0, 1, 0) = 0$$

$$\Rightarrow (a_1 + a_2 + a_3 + 0a_4, 0a_1 + a_2 + a_3 + a_4, 0a_1 + 0a_2 + a_3 + 0a_4) = (0, 0, 0)$$

$$\Rightarrow a_1 + a_2 + a_3 = 0, a_2 + a_3 + a_4 = 0, a_3 = 0$$

The coefficient matrix of these equations are in the matrix form.

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Here,

$$|A| = 1(1 - 0) = 1 \neq 0$$

$$\Rightarrow R(A) = 3$$

$\therefore$  The scalars  $a_1, a_2, a_3, a_4$  are not all zero, therefore, the given set  $S$  of vectors is linearly dependent and hence  $S$  is not a basis set. Proved.

**Example 33.** Show that the set  $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1), (0, 1, 0)\}$  is not a basis set.

## 34.1 LINEAR TRANSFORMATIONS

Let  $U$  and  $V$  be two vector spaces over the same field  $F$ . Then, a mapping  $T$  of  $U$  into  $V$  is called a linear transformation or a homomorphism of  $U$  into  $V$ , if

$$(i) T(\alpha + \beta) = T(\alpha) + T(\beta) \quad \forall \alpha, \beta \in U \text{ and } (ii) T(a\alpha) = aT(\alpha) \quad \forall a \in F, \alpha \in U.$$

The conditions (i) and (ii) above can be combined as

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) \quad \forall a, b \in F, \forall \alpha, \beta \in U$$

A transformation of  $U$  into itself is called a linear operator.

A one-one linear transformation of  $U$  onto  $V$  is called an isomorphism. In case, there exist an isomorphism of  $U$  onto  $V$ , we say that  $U$  is isomorphic to  $V$  and we write,  $U \cong V$ .

(i) **Zero Transformation.** If  $U(F)$  and  $V(F)$  be two vector spaces over the same field  $F$ , then the mapping  $\hat{0}: U \rightarrow V$  defined by  $\hat{0}(x) = 0, \forall x \in U$  is said to be **zero transformation**.  $\hat{0}$  is called **zero operator**.

(ii) **Identity Transformation (or Identity operator).** If  $V(F)$  is a vector space, then the mapping  $T: V \rightarrow V$  defined by  $T(v) = v \quad \forall v \in V$  is called an **identity transformation**.  $T$  is called **identity operator**.

**Theorem 1.** To prove that zero operator is linear operator.

**Proof.** Let  $U(F)$  and  $V(F)$  be two vector spaces over the same field  $F$ . Let  $\hat{0}: U \rightarrow V$  defined by  $\hat{0}(x) = 0 \quad \forall x \in U$  be the zero operator.

Let  $\alpha, \beta \in U$  and  $a, b \in F$ , then  $\hat{0}(a\alpha + b\beta) = 0 = 0 + 0 = a\hat{0}(\alpha) + b\hat{0}(\beta)$ .

Hence,  $\hat{0}$  is a linear operator.

**Proved.**

**Theorem 2.** To prove that identity operator is linear operator.

**Proof.** Let  $T$  be the identity operator on  $V(F)$ . Then  $T(x) = x, \forall x \in V$ .

Let  $\alpha, \beta \in V$  and  $a, b \in F$ , then  $a\alpha + b\beta \in V$

$$\begin{aligned} \text{Now, } T(a\alpha + b\beta) &= a\alpha + b\beta && \{\text{by definition of } T\} \\ &= aT(\alpha) + bT(\beta) && [\because \alpha = T(\alpha), \beta = T(\beta)] \end{aligned}$$

Hence,  $T$  is a linear operator.

**Proved.**

**Example 1.** Show that the translation mapping  $f: V_2(R) \rightarrow V_2(R)$  defined by

$$f(x, y) = (x + 2, y + 3) \text{ is not linear.}$$

**Solution.** Here  $\bar{0} = (0, 0)$  is the zero vector of  $V_2(R)$ . Thus, by definition of  $f$ , we have  
 $f(\bar{0}) = f(0, 0) = (0+2, 0+3) = (2, 3) \neq 0$ .  
Since  $f$  does not map the zero vector onto the zero vector, hence  $f$  is not linear.  
Proved

**Example 2.** Prove that:  $T: R^2 \rightarrow R^2$   
 $T(x_1, x_2) = (x_1, 0)$ , is a linear transformation.

**Solution.** Let  $a, b \in R$  and  $x = (x_1, x_2)$  and  $y = (y_1, y_2) \in R^2$

$$\begin{aligned} \text{We see } T &= T\{a(x_1, x_2) + b(y_1, y_2)\} \\ &= T\{(ax_1, ax_2) + (by_1, by_2)\} \\ &= T\{(ax_1 + by_1, ax_2 + by_2)\} \\ &= (ax_1 + by_1, 0) \\ &= (ax_1, 0) + (by_1, 0) \\ &= a(x_1, 0) + b(y_1, 0) \\ &= aT(x_1) + bT(y_1). \end{aligned}$$

Proved

So,  $T$  is a linear transformation.

**Example 3.** Show that the mapping  $f: V_2(R) \rightarrow V_3(R)$  defined by

$f(a, b) = (a, b, 0)$  is a linear transformation. (R.G.P.V. Bhopal, III Semester, Dec. 2007)

**Solution.** Let  $\alpha = (a_1, b_1), \beta = (a_2, b_2) \in V_2(R)$

$$\begin{aligned} \text{If } a, b \in F, \text{ then } f(a\alpha + b\beta) &= f[a(a_1, b_1) + b(a_2, b_2)] \\ &= f[(aa_1, ab_1) + (ba_2, bb_2)] \\ &= f(aa_1 + ba_2, ab_1 + bb_2) \\ &= (aa_1 + ba_2, ab_1 + bb_2, 0) \\ &= a(a_1, b_1, 0) + b(a_2, b_2, 0) \\ &= af(\alpha) + bf(\beta) \end{aligned}$$

Proved

Hence,  $f$  is a linear transformation.

**Example 4.** Show that the mapping  $f: V_3(R) \rightarrow V_2(R)$  defined by  $f(a, b, c) = (c, a+b)$  is a linear transformation.

**Solution.** Let  $\alpha = (a_1, b_1, c_1), \beta = (a_2, b_2, c_2) \in V_3(R)$ . If  $a, b, c \in R$ , then

$$\begin{aligned} f(a\alpha + b\beta) &= f[a(a_1, b_1, c_1) + b(a_2, b_2, c_2)] = f[(aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2)] \\ &= (ac_1 + bc_2, aa_1 + ba_2 + ab_1 + bb_2) \quad [\text{By definition of } f] \\ &= (ac_1, aa_1 + ab_1) + (bc_2, ba_2 + bb_2) = a(c_1, a_1 + b_1) + b(c_2, a_2 + b_2) \\ &= af(\alpha) + bf(\beta) \end{aligned}$$

Proved

Hence,  $f$  is a linear transformation.

**Example 5.** Show that the mapping  $f: V_2(R) \rightarrow V_2(R)$  defined by  $f(x, y) = (x^3, y^3)$  is not a linear transformation.

**Solution.** Let  $\alpha = (x_1, y_1), \beta = (x_2, y_2) \in V_2(R)$ , then

$$\begin{aligned} f(\alpha + \beta) &= f[(x_1, y_1) + (x_2, y_2)] = f[(x_1 + x_2, y_1 + y_2)] = [(x_1 + x_2)^3, (y_1 + y_2)^3] \\ &\neq (x_1^3 + x_2^3, y_1^3 + y_2^3) \\ &\neq (x_1^3, y_1^3) + (x_2^3, y_2^3) \\ &\neq f(\alpha) + f(\beta) \end{aligned}$$

Hence,  $f$  is not a linear transformation.

Proved

**Example 6.** Show that the mapping  $f: V_3(R) \rightarrow V_2(R)$  defined by  $f(a, b, c) = (a - b, a + c)$  is linear transformation. (R.G.P.V., Bhopal, III Semester, Dec. 2006)

**Solution.** Let  $\alpha, \beta \in V_3$

$$\alpha = (a_1, b_1, c_1), \quad \beta = (a_2, b_2, c_2)$$

$$\begin{aligned} f(\alpha + \beta) &= f[(a_1, b_1, c_1) + (a_2, b_2, c_2)] \\ &= f[(a_1 + a_2, b_1 + b_2, c_1 + c_2)] \\ &= f[a_1 + a_2 - b_1 + b_2, a_1 + a_2, c_1 + c_2] \\ &= f[a_1 - b_1 + a_2 + b_2 + a_1 + c_1, a_2 + c_2] \\ &= (a_1 - b_1, a_1 + c_1) + (a_2 + b_2, a_2 + c_2) \\ &= f(a_1, b_1, c_1) + f(a_2, b_2, c_2) \\ &= f(\alpha) + f(\beta) \end{aligned}$$

For any real number

$$\begin{aligned} f(k\alpha) &= f[k(a_1, b_1, c_1)] \\ &= f(ka_1, kb_1, kc_1) \\ &= (ka_1 - kb_1, ka_1 + kc_1) \\ &= k(a_1 - b_1, a_1 + c_1) \\ &= kf(a_1, b_1, c_1) \\ &= kf(\alpha) \end{aligned}$$

So,  $f$  is a linear transformation from  $V_3$  to  $V_2$ .

**Proved.**

**Example 7.** Show that the function  $T: V_2 \rightarrow V_2$  defined by  $T(x, y) = (x^2, y^2)$  is not a linear transformation.

## 34.2 MATRIX OF A LINEAR TRANSFORMATION

Consider the simultaneous equations given below:

$$\begin{aligned}2x_1 + 3x_2 - x_3 &= 2 \\5x_1 - 6x_2 - 3x_3 &= 10 \\x_1 + x_2 + x_3 &= 8\end{aligned}$$

The left hand side of the equations can be considered as the linear transformations of  $T$

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 - x_3 \\ 5x_1 - 6x_2 - 3x_3 \\ x_1 + x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 5 & 6 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

we can write the formula  $A: T_A(X) = AX$

In general for  $m \times n$  matrix the transformation is  $TA : R^n \rightarrow R^m$   
such transformation is called matrix transformation.

For example, the matrix  $\begin{bmatrix} 1 & 5 & 3 \\ 2 & 7 & 9 \\ 4 & 1 & 2 \end{bmatrix}$  gives matrix transformation

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 5x_2 + 3x_3 \\ 2x_1 + 7x_2 + 9x_3 \\ 4x_1 + x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 7 & 9 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

This can also be written horizontally

$$T : R^3 \rightarrow R^3, [x_1, x_2, x_3] \rightarrow [x_1 + 5x_2 + 3x_3, 2x_1 + 7x_2 + 9x_3, 4x_1 + x_2 + 2x_3]$$

This transformation is not matrix transformation because it can not be expressed as  $A X$  for constant matrix  $A$ .

**Example 11.** The matrix of linear mapping  $T : R^3 \rightarrow R^3$  given by  $T(a, b, c) = (a, b, 0)$  relative to standard basis is

$$(i) \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \quad (ii) \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \quad (iii) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (iv) \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(R.G.P.V., Bhopal, III Semester, Dec. 2006)

**Solution.** The standard basis of  $R^3$  is  $B = (e_1, e_2, e_3)$ , where  $e_1 = (1, 0, 0)$ ,  $e_2 = (0, 1, 0)$  and  $e_3 = (0, 0, 1)$

Thus by definition of  $T$ , we have

$$\begin{aligned} T(e_1) &= T(1, 0, 0) = (1, 0, 0) = 1(1, 0, 0) + 0(0, 1, 0) + 0(0, 0, 1) \\ &= 1e_1 + 0e_2 + 0e_3 \end{aligned}$$

$$\begin{aligned} T(e_2) &= T(0, 1, 0) = (0, 1, 0) = 0(1, 0, 0) + 1(0, 1, 0) + 0(0, 0, 1) \\ &= 0e_1 + 1.e_2 + 0.e_3 \end{aligned}$$

$$\begin{aligned} T(e_3) &= T(0, 0, 1) = (0, 0, 1) = 0(1, 0, 0) + 0(0, 1, 0) + 1(0, 0, 1) \\ &= 0e_1 + 0e_2 + e_3 \end{aligned}$$

Hence, the coefficient matrix is  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  and its transpose

matrix is  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$[T, B] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ans. (iii) is correct

## Function or Mapping

Let there be two non-empty sets  $X$  and  $Y$  and there is some rule or correspondence which assigns to each element  $x \in X$ , a unique element  $y \in Y$ , then this rule or correspondence which is said to be a *mapping* or a *function* and denoted by  $f$ , i.e.,  $f: X \rightarrow Y$  and read as ' $f$  is a function of  $X$  to  $Y$ ' or  $f$  is a 'mapping of  $X$  to  $Y$ '.

The set  $X$  is called the *domain* of the given function  $f$  and the set  $Y$  of all the values assumed by it is called its *Range* or *Image set*. Also  $Y$  is called the *co-domain* of  $f$ .

$y$  is sometimes known as image of  $x$  and written as  $y = f(x)$ . Here  $f(x)$  is read as 'image of  $x$  under the rule  $f$ ' or simply ' $f$  of  $x$ '. The rule  $f$  is also known as *mapping* or *transformation* or *operator* and  $x$  is also known as *pre-image* of  $y$ .

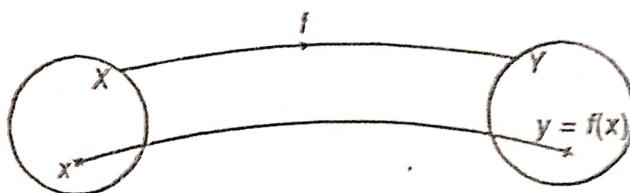


Fig. 4.3

A function whose range has a single element is said to be *constant function*.

Diagrammatical representation of  $y = f(x)$  with a rule  $f$  defined by  $x \rightarrow f(x)$  is shown in Fig. 4.3.

(If  $y = x^2$ , then the rule  $f$  is  $x \rightarrow x^2$   
which is shown in Fig. 4.4 for positive integral values of  $x$ .



Fig. 4.4

**Functions defined as sets of ordered pairs.** Given two non-empty sets  $X$  and  $Y$ , a function  $f$  from  $X$  to  $Y$  is a subset of  $X \times Y$  provided

- (i)  $\forall x \in X, (x, y) \in f$  for some  $y \in Y$ , i.e.,  $\exists$  (there exists) a rule  $f$  so that every element of  $X$  has image.
- (ii)  $(x, y) \in f$  and  $(x, y') \in f \Rightarrow y = y'$ , i.e., the image is unique.

The graph of  $f$  is defined as the subset of  $X \times Y$  given by  $\{(x, f(x)) : x \in X\}$ , and that range of  $f$  as the set of all images under  $f$  given by  $f[X] = \{y \in Y : y = f(x) \text{ for some } x \in X\} = \{f(x) : x \in X\}$ .

In case  $A \subset X$  then the set  $\{f(x) : x \in A\}$  is known as the *image* of  $A$  under  $f$  and denoted by  $f[A]$ . Also if  $B \subset Y$ , then the set  $\{x \in X : f(x) \in B\}$  is known as the *inverse image* of  $B$  under  $f$  and denoted by  $f^{-1}[B]$ .

**Extension and Restriction of a function.** Given two functions  $f$  and  $g$  such that  $f$  contains the domain of  $g$  and  $f(x) = g(x) \forall x$  in the domain of  $g$ , the function  $f$  is said to be the *extension* of  $g$  and  $g$  is said to be the *restriction* of  $f$ .

**Real and Complex functions.** If range of  $f$  consists of real numbers,  $f$  is said to be a *real function* and if its range consists of complex numbers,  $f$  is said to be a *complex function*.

**Onto and Into Mappings.** If the range is completely filled up, the mapping is said to be *onto* and if the range is not completely filled up then it is *into*. In other words, if  $\exists$  at least one  $y \in Y$  which is not an  $f(x)$  for any  $x \in X$ , then the mapping  $f$  is said to be *into* otherwise it is said to be *onto* or *surjective*. The surjective function is also known as a *surjection* or an *epimorphism*.

**One-one and Many-one Mappings.** Given two non-empty sets  $X$  and  $Y$ , if two different elements in  $X$  always have different images under the rule  $f$ , then  $f$  is said to be a *one-one mapping* or an *injection* or *monomorphism* of  $X$  into (onto)  $Y$  and if the two or more different elements of  $X$  have the same image under  $f$ , then  $f$  is said to be a *many-one mapping* of  $X$  into (onto)  $Y$ .

Diagrammatical representation of such functions are shown in Figs. 4.5, 4.6, 4.7, 4.8.

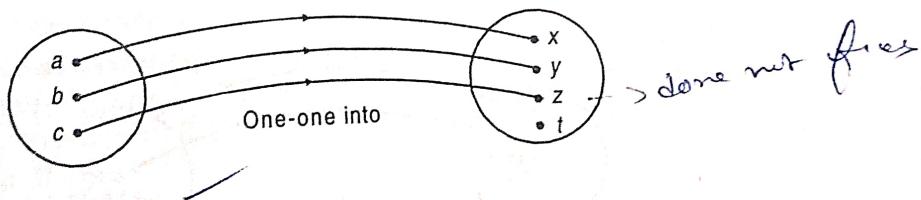


Fig. 4.5

A function which is both surjective and injective is known as *bijection*, i.e., a one-one onto mapping is also known as a *bijection* and a bijection of a set  $X$  onto itself is known as *Permutation of  $X$* .

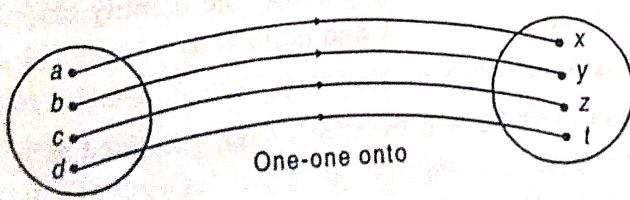


Fig. 4.6

If  $f: X \rightarrow Y$ ,  $f$  is one-one if  $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \forall x_1, x_2 \in X$ . In case  $f$  is *into*, the range of  $f$  is proper subset of  $Y$ , i.e.,  $f[X] \subset Y$  and  $f[X] \neq Y$ .

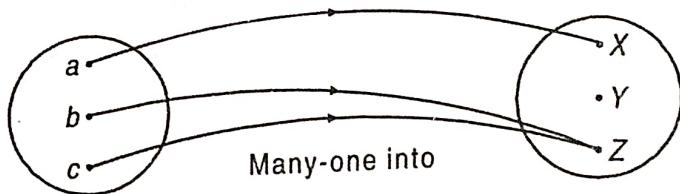


Fig. 4.7

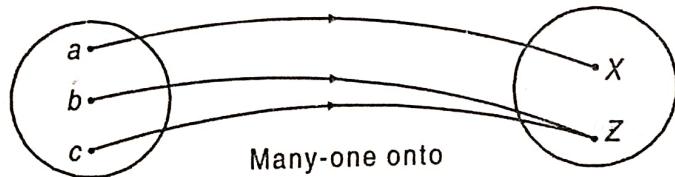


Fig. 4.8

In case  $f$  is *onto*, the range of  $f$  is equal to  $Y$ , i.e.,  $f[X] = Y$ .

**Inverse mapping.** Let  $f$  represent a function (mapping) which is both *onto* and *one-one* defined as  $f: X \rightarrow Y$ , then its inverse mapping  $f^{-1}: Y \rightarrow X$  is defined as below:

$\forall y \in Y$ , if we find the unique element  $x \in X$  s.t.  $f(x) = y$  then  $x$  is defined to be  $f^{-1}(y)$ , i.e.,  $f^{-1}(y) = \{x : x \in X, f(x) = y\}$  which follows that  $f^{-1}(y)$  is always a subset of  $X$ .

Diagrammatical representation of an inverse mapping is shown in Fig. 4.9.

One-one onto mapping is often called as *one-to-one correspondence*. Thus if  $f$  is a one-to-one correspondence between  $X$  and  $Y$ , then  $f^{-1}$  is a one-to-one correspondence between  $Y$  and  $X$ .

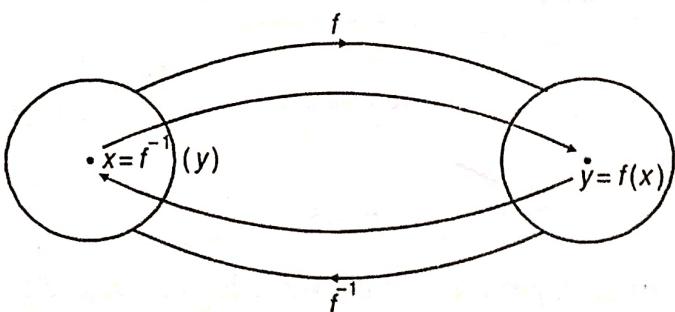


Fig. 4.9